On tight spherical designs.

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ABSTRACT: Let X be a tight t-design of dimension n for one of the open cases $t = 5$ or $t = 7$. An investigation of the lattice generated by X using arithmetic theory of quadratic forms allows to exclude infinitely many values for n.

1 Introduction.

Spherical designs have been introduced in 1977 by Delsarte, Goethals and Seidel [5] and shortly afterwards studied by Eiichi Bannai in a series of papers (see [1], [2], [3] to mention only a few of them). A spherical t-design is a finite subset X of the sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid (x, x) = 1\}$$

such that every polynomial on $\mathbb{R}^n$ of total degree at most t has the same average over X as over the entire sphere. Of course the most interesting t-designs are those of minimal cardinality. If $t = 2m$ is even, then any spherical t-design $X \subset S^{n-1}$ satisfies

$$|X| \geq \binom{n-1+m}{m} + \binom{n-2+m}{m-1}$$

and if $t = 2m + 1$ is odd then

$$|X| \geq 2\binom{n-1+m}{m}.$$

A t-design X for which equality holds is called a **tight** t-design.

Tight t-designs in $\mathbb{R}^n$ with $n \geq 3$ are very rare. In [1] and [2] it is shown that such tight designs only exist if $t \leq 5$ and $t = 7, 11$. The tight t-designs with $t = 1, 2, 3$ as well as $t = 11$ are completely classified whereas their classification for $t = 4, 5, 7$ is still an open problem. It is known that the existence of a tight 4-design in dimension $n - 1$ is equivalent to the existence of a tight 5-design in dimension n, so the open cases are $t = 5$ and $t = 7$. It is also well known that tight spherical t-designs X for odd values of t are antipodal, i.e. $X = -X$ (see [5]).

There are certain numerical conditions on the dimension of such tight designs. A tight 5-design $X \subset S^{n-1}$ can only exist if either $n = 3$ and X is the set of 12 vertices of a regular icosahedron or $n = (2m + 1)^2 - 2$ for an integer m ([5], [1], [2]). Existence is only known for $m = 1, 2$ and these designs are unique. Using lattices [4] excludes the next two open cases $m = 3, 4$ as well as an infinity of other values of m. Here we exclude infinitely many other cases including $m = 6$.

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There are similar results for tight 7-designs. Such designs only exist if $n = 3d^2 - 4$ where the only known cases are $d = 2, 3$ and the corresponding designs are unique. The paper [4] excludes the cases $d = 4, 5$ and also gives partial results on the interesting case $d = 6$ which still remains open. For odd values of d we use characteristic vectors of the associated odd lattice of odd determinant to show that d is either $\pm 1 \pmod{16}$ or $\pm 3 \pmod{32}$ (see Theorem 3.5). We also exclude infinitely many even d in Theorem 3.3.

2 General equalities.

We always deal with antipodal sets and write them as disjoint union

$$X \mathbin{\dot\cup} -X \subset S^{n-1}(d) = \{x \in \mathbb{R}^n \mid (x,x) = d\} \text{ with } s := |X| \in \mathbb{N}.$$

By the theory developed in [7] the set $X \mathbin{\dot\cup} -X$ is a 7-design if and only if for all $\alpha \in \mathbb{R}^n$

$$(D6)(\alpha): \quad \sum_{x \in X} (x,\alpha)^6 = \frac{3 \cdot 5 s d^3}{n(n+2)(n+4)} (\alpha,\alpha)^3.$$

Applying the Laplace operator to $(D6)(\alpha)$ one obtains

$$(D4)(\alpha): \quad \sum_{x \in X} (x,\alpha)^4 = \frac{3sd^2}{n(n+2)} (\alpha,\alpha)^2 \text{ and}$$

$$(D2)(\alpha): \quad \sum_{x \in X} (x,\alpha)^2 = \frac{sd}{n} (\alpha,\alpha).$$

Substituting $\alpha = \sum_{i=1}^6 \xi_i \alpha_i$ in $(D6)$ $(D4)$ and $(D2)$ we find that for all $\alpha, \beta \in \mathbb{R}^n$:

$$\begin{array}{lll}
(D11) & \sum_{x\in X}(x,\alpha)(x,\beta) & = \frac{sd}{n}(\alpha,\beta) \\
(D13) & \sum_{x\in X}(x,\alpha)(x,\beta)^3 & = \frac{3sd^2}{n(n+2)}(\alpha,\beta)(\beta,\beta) \\
(D22) & \sum_{x\in X}(x,\alpha)^2(x,\beta)^2 & = \frac{sd^2}{n(n+2)}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \\
(D15) & \sum_{x\in X}(x,\alpha)(x,\beta)^5 & = \frac{3 \cdot 5sd^3}{n(n+2)(n+4)}(\beta,\beta)^2(\alpha,\beta) \\
(D24) & \sum_{x\in X}(x,\alpha)^2(x,\beta)^4 & = \frac{3sd^3}{n(n+2)(n+4)}((\beta,\beta)^2(\alpha,\alpha) + 4(\alpha,\beta)^2(\beta,\beta)) \\
(D33) & \sum_{x\in X}(x,\alpha)^3(x,\beta)^3 & = \frac{3sd^3}{n(n+2)(n+4)}(2(\alpha,\beta)^3 + 3(\alpha,\alpha)(\beta,\beta)(\alpha,\beta))
\end{array}$$

Similarly $X \mathbin{\dot\cup} -X$ is a spherical 5-design, if an only if $(D4)$ and $(D2)$ hold for any $\alpha \in \mathbb{R}^n$. Then we obtain the equations $(D11)$, $(D13)$. and $(D22)$.

We will consider the lattice $\Lambda := \langle X \rangle$ and $\alpha \in \Lambda^*$. Then (α, x) is integral for all $x \in X$. This yields certain integrality conditions for the norms and inner products of elements in Λ^*:

Lemma 2.1 *If $X \mathbin{\dot\cup} -X \subset S^{n-1}(d)$ is a spherical 5-design then for all $\alpha, \beta \in \Lambda^*$*

$$\frac{sd}{12n}(\alpha,\alpha)(\frac{d}{n+2}(\alpha,\alpha) - 1) \in \mathbb{Z}$$

and

$$\frac{sd}{6n}(\alpha,\beta)(\frac{d}{n+2}(\alpha,\alpha) - 1) \in \mathbb{Z}$$

Proof. Let $x \in X$ and $k := (x, \alpha)$. Then $k^4 - k^2$ is a multiple of 12 and hence $\frac{1}{12}\sum_{x\in X}(x,\alpha)^4 - (x,\alpha)^2 \in \mathbb{Z}$ which yields the first divisibility condition. Similarly $k^3 - k$ is a multiple of 6 and so

$$\frac{1}{6}\sum_{x\in X}(x,\beta)((x,\alpha)^3 - (x,\alpha)) = \frac{1}{6}(D13 - D11) \in \mathbb{Z}$$

$\square$

Similarly

$$(\beta, x)(\alpha, x)((\alpha, x)^2 - 1)((\alpha, x)^2 - 4) = (\beta, x)(\alpha, x)^5 - 5(\beta, x)(\alpha, x)^3 + 4(\beta, x)(\alpha, x)$$

is divisible by 5 consecutive integers and hence this quantity is a multiple of 120 for any $\alpha, \beta \in \Lambda^*$ and $x \in X$.

Moreover $(\alpha, x)((\alpha, x)^2 - 1)$ is divisible by 3 consecutive integers and therefore a multiple of 6, hence

$$(\beta, x)((\beta, x)^2 - 1)(\alpha, x)((\alpha, x)^2 - 1) = (\beta, x)(\alpha, x)((\beta, x)^2(\alpha, x)^2 - (\beta, x)^2 - (\alpha, x)^2 + 1)$$

is divisible by 36. Summing over all $x \in X$ we obtain that the right hand side of $D15 - 5D13 + 4D11$ is a multiple of 120 and that $D33 - D13 - D31 + D11$ is divisible by 36.

Lemma 2.2 *If $X \dot\cup -X \subset S^{n-1}(d)$ is a spherical 7-design then for all $\alpha, \beta \in \Lambda^*$*

$$\frac{1}{120}(\alpha,\beta)(\frac{3\cdot 5sd^2}{n(n+2)}(\alpha,\alpha)(\frac{d}{n+4}(\alpha,\alpha) - 1) + 4\frac{sd}{n}) \in \mathbb{Z}$$

and

$$\frac{1}{36}(\alpha,\beta)(\frac{3sd^2}{n(n+2)}(\frac{d}{n+4}(2(\alpha,\beta)^2 + 3(\alpha,\alpha)(\beta,\beta)) - (\alpha,\alpha) - (\beta,\beta)) + \frac{sd}{n}) \in \mathbb{Z}.$$

3 Tight spherical 7-designs.

Let $X \dot\cup -X \subset S^{n-1}(d)$ be a tight spherical 7-design. Then $n = 3d^2 - 4$, $(x, y) \in \{0, \pm 1\}$ for all $x \neq y \in X$ and $s := |X| = n(n+1)(n+2)/6$.

Let $\Lambda = \langle X \rangle$ be the lattice generated by the set X and put $\Gamma := \Lambda^*$. Then Λ is an integral lattice and Λ is even, if d is even. Substituting these values into the formulas of Lemma 2.2 we obtain

Lemma 3.1 *For all $\alpha, \beta \in \Gamma$ we have*

$$((d^3 - d)/240)(\alpha,\beta)(12d^2 - 8 - 15d(\alpha,\alpha) + 5(\alpha,\alpha)^2) \in \mathbb{Z}$$

and

$$((d^3 - d)/72)(\alpha,\beta)(3(\alpha,\alpha)(\beta,\beta) - 3d((\alpha,\alpha) + (\beta,\beta)) + 2(\alpha,\beta)^2 + (3d^2 - 2)) \in \mathbb{Z}.$$

For a prime p let v_p denote the p-adic valuation on $\mathbb{Q}$.

Corollary 3.2 *(improvement of [4, Lemma 4.2])*

(i) Let $p \geq 5$ be a prime. If $v_p(d^3 - d) \leq 2$ then $v_p((\alpha,\alpha)) \geq 0$ for all $\alpha \in \Gamma$.

(ii) If $v_3(d^3 - d) \leq 4$ then $v_3((\alpha,\alpha)) \geq 0$ for all $\alpha \in \Gamma$.

(iii) If $v_2(d^3 - d) \leq 6$ then $v_2((\alpha,\alpha)) \geq 0$ for all $\alpha \in \Gamma$.

(iv) If d is even but not divisible by 8 then $v_2((\alpha,\alpha)) \geq 1$ for all $\alpha \in \Gamma$.

(v) If d is even but not divisible by 32 then $v_2((\alpha,\beta)) \geq 0$ for all $\alpha,\beta \in \Gamma$.

(vi) If d is odd and $v_2(d^2 - 1) \leq 4$ then $v_2((\alpha,\beta)) \geq 0$ for all $\alpha,\beta \in \Gamma$.

Proof. Part (i),(iii) and (iv) are the same as in [4, Lemma 4.2] and follow from the first congruence in Lemma 3.1.
For (ii) we use the second congruence in the special case $\alpha = \beta$. Under the assumption we obtain $v_3((d^3 - d)/72) \leq 4 - 2 \leq 2$. If $v_3((\alpha,\alpha)) \leq -1$ then

$$v_3(5(\alpha,\alpha)^3 - 6d(\alpha,\alpha)^2 + (3d^2 - 2)(\alpha,\alpha)) = v_3((\alpha,\alpha)^3) \leq -3$$

contradicting the fact that the product is integral.
To see (v) we use (iii) to see that $v_2((\alpha,\alpha)) \geq 0$ for all $\alpha \in \Gamma$. Then the second congruence yields that $v_2(\frac{d}{4}(\alpha,\beta)^3) \geq 0$. Since $v_2(d) < 5$ we obtain $v_2((\alpha,\beta)) \geq 0$.
The last assertion (vi) is obtained by the same argument. $\square$

Using this observation we can extend [4, Theorem 4.3] which only treats the case $v_2(d) = 2$.

Theorem 3.3 *Assume that $v_p(d^3 - d) \leq 2$ for all primes $p \geq 5$ and that $v_3(d^3 - d) \leq 4$. If $v_2(d) = 2, 3$ or 4 then a tight spherical 7-design in dimension $n = 3d^2 - 4$ does not exist.*

Proof. Γ is integral by Corollary 3.2 and therefore Λ is an even unimodular lattice of dimension $n \equiv 4 \pmod 8$ which gives a contradiction. $\square$

A similar argument allows to deduce the following lemma from Corollary 3.2.

Lemma 3.4 *If d is odd and $v_2(d^2 - 1) \leq 4$ then Λ is an odd lattice of odd determinant. If additionally $v_p(d^3 - d) \leq 2$ for all primes $p \geq 5$ and $v_3(d^3 - d) \leq 4$ then $\Lambda = \Lambda^*$ is an odd unimodular lattice.*

In particular if d is odd and $d \not\equiv \pm 1 \pmod{16}$ then Λ is an odd lattice of odd determinant. Over the 2-adic numbers there is an orthogonal basis

$$\Lambda \otimes \mathbb{Z}_2 \cong \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}_2} \text{ with } (b_i, b_j) = 0, (b_k, b_k) = 1, (b_n, b_n) = 1 + \delta \in \{1, 3, 5, 7\}$$

for $1 \leq i \neq j \leq n$, $k = 1, \ldots, n-1$. Such a lattice contains characteristic vectors. These are elements $\alpha \in \Lambda \otimes \mathbb{Z}_2$ such that

$$(\alpha, \lambda) \equiv (\lambda, \lambda) \pmod 2, \text{ for all } \lambda \in \Lambda \otimes \mathbb{Z}_2.$$

Using the basis above, the characteristic vectors in Λ are the vectors

$$\alpha = \sum_{i=1}^{n} a_i b_i \text{ with } a_i \in 1 + 2\mathbb{Z}_2 \text{ of norm } (\alpha, \alpha) \equiv n + \delta \pmod 8.$$

Theorem 3.5 *Let $X \dot\cup -X$ be a tight 7-design of dimension $3d^2 - 4$ with odd d. Assume that $d \not\equiv \pm 1 \pmod{16}$. Then either $d \equiv 3 \pmod{32}$ and $\det(\Lambda) \in (\mathbb{Z}_2^*)^2$ or $d \equiv -3 \pmod{32}$ and $\det(\Lambda) \in 3(\mathbb{Z}_2^*)^2$. If additionally $v_p(d^3 - d) \leq 2$ for all primes $p \geq 5$ and $v_3(d^3 - d) \leq 4$ then $d \not\equiv -3 \pmod{16}$.*

Proof. Let $\Lambda = \langle X \rangle_{\mathbb{Z}_2}$ and $\alpha \in \Lambda$ be a characteristic vector of Λ of norm $(\alpha, \alpha) = n + \delta - 8a$ for some $a \in \mathbb{Z}_2$ and $\delta \in \{0, 2, 4, 6\}$. Then $(\alpha, \lambda) \equiv (\lambda, \lambda) \pmod 2$ for all $\lambda \in \Lambda$, in particular (α, x) is odd for all $x \in X$. For $k > 0$ let

$$n_k := |\{x \in X \mid (x, \alpha) = \pm k\}|.$$

Then from (D2), (D4), (D6) we obtain

$$\begin{array}{lll}
(D0) & \sum n_k & = |X| = (1/2)(3d^2 - 4)(3d^2 - 2)(d^2 - 1) \\
(D2) & \sum k^2 n_k & = (1/2)(3d^2 - 2)(d^2 - 1)d(n + \delta - 8a) \\
(D4) & \sum k^4 n_k & = (3/2)(d^2 - 1)d^2(n + \delta - 8a)^2 \\
(D6) & \sum k^6 n_k & = (5/2)(d^2 - 1)d(n + \delta - 8a)^3.
\end{array}$$

Now $n_k \neq 0$ only for odd k. If k is odd, then $(k^2 - 1)$ is a multiple of 8 and $(k^2 - 1)(k^2 - 9)$ is a multiple of $8 \cdot 16$. Now $(k^2 - 1)(k^2 - 9)(k^2 - 25) = k^6 - 35k^4 + 259k^2 - 225$ is a multiple of $2^{10}3^2 5$ in particular

$$(a) \;\; 2^{-7}((D4) - 10(D2) + 9(D0)) \in \mathbb{Z}.$$

and

$$(b) \;\; 2^{-10}((D6) - 35(D4) + 259(D2) - 225(D0)) \in \mathbb{Z}.$$

We substitute $d = 16b + r$ for $r = \pm 3, \pm 5, \pm 7$ into these congruences to obtain polynomials in a where the coefficients are polynomials in b. The contradictions we obtain in the respective cases are listed below the table.

$r =$	3	5	7	-7	-5	-3
$\delta = 0$	$(c0)$	$(a2)$	$(b1)$	$(a1)$	$(c2)$	$(a2)$
$\delta = 2$	$(a2)$	$(c2)$	$(a1)$	$(b1)$	$(a2)$	$(c0)$
$\delta = 4$	$(c1)$	$(a2)$	$(b2)$	$(a1)$	$(c1)$	$(a2)$
$\delta = 6$	$(a2)$	$(c1)$	$(a1)$	$(b2)$	$(a2)$	$(c1)$

(a) In congruence (a) the coefficients of a and a^2 are in $\mathbb{Z}[b]$ but the constant coefficient is

  (a1) $p(b) + \frac{b}{2} + \frac{x}{4}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

  (a2) $p(b) + \frac{b}{2} + \frac{x}{8}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

(b) In congruence (b) the coefficients of a, a^2 and a^3 are in $\mathbb{Z}[b]$ but the constant coefficient is

(b1) $p(b) + \frac{1}{2}$ with $p(b) \in \mathbb{Z}[b]$.

(b2) $p(b) + \frac{b}{2} + \frac{x}{4}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

(c) In congruence (b) the coefficient of a^3 is in $\mathbb{Z}[b]$ the ones of a and a^2 are in $\frac{1}{2} + \mathbb{Z}[b]$ but the constant coefficient is

(c0) $p(b) + \frac{b}{2}$ with $p(b) \in \mathbb{Z}[b]$. Here we can only deduce that b is even.

(c1) $p(b) + \frac{x}{8}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

(c2) $p(b) + \frac{b}{2} + \frac{x}{4}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

Hence only the cases $r = 3$, $\delta = 0$ and $r = -3$, $\delta = 2$ are possible and then b is even.
□

To summarize we list a few small values that are excluded by Theorem 3.5 and Theorem 3.3:

Corollary 3.6 *There is no tight 7-design of dimension $n = 3d^2 - 4$ for*

$$d \in \{4, 5, 7, 8, 9, 11, 12, 13, 16, 19, 20, 21, \ldots\}$$

4 Tight spherical 5-designs.

Assume that $d = 2m + 1$ and that $X \dot{\cup} -X$ is a tight spherical 5-design in dimension $n = d^2 - 2$. Then $|X| = n(n+1)/2$ and scaled such that $(x, x) = d$ for all $x \in X$ we have $(x, y) = \pm 1$ for $x \neq y \in X$ and $\Lambda := \langle X \rangle$ is an odd integral lattice. With these values the formula $(D4)$ reads as

$$(D4) \quad \sum_{x \in X} (x, \alpha)^4 = 6m(m+1)(\alpha, \alpha)^2.$$

Lemma 4.1 *(see [4, Lemma 3.6]) Assume that $m(m + 1)$ is not divisible by the square of a prime $p \geq 5$. Then $(\alpha, \alpha) \in \mathbb{Z}[1/6]$ for all $\alpha \in \Lambda^*$.*

Substituting the special values into the formula of Lemma 2.1 we immediately obtain

Lemma 4.2 *(see [4, Lemma 3.3]) For all $\alpha \in \Lambda^*$*

$$\frac{1}{6} m(m+1)(\alpha, \alpha)(3(\alpha, \alpha) - (2m + 1)) \in \mathbb{Z}$$

Corollary 4.3 *If $m(m + 1)$ is not a multiple of 8, then $(\alpha, \alpha) \in \mathbb{Z}_2$ is 2-integral for all $\alpha \in \Lambda^*$.*

We now treat the Sylow 3-subgroup $D_3 := \mathrm{Syl}_3(\Lambda^*/\Lambda)$.

Lemma 4.4 *Assume that $m(m+1)$ is not a multiple of 9. Then $|D_3| \in \{1,3\}$.*

Proof. Assume that $D_3 \neq 1$. Since D_3 is a regular quadratic 3-group it contains an anisotropic element $\alpha + \Lambda \in \Lambda^*/\Lambda$ with $(\alpha,\alpha) = \frac{p}{q}$ and $3 \mid q$. By equality $(D4)$ the denominator q is not divisible by 9, in particular the exponent of D_3 is 3 and $(\alpha,\alpha) = \frac{p}{3}$ with a 3-adic unit $p \equiv \pm 1 \pmod 3$. Now Lemma 4.2 gives

$$\frac{1}{18} m(m+1)p(p-(2m+1)) \in \mathbb{Z}$$

Since $m(m+1)$ is not a multiple of 9, this implies that $p \equiv (2m+1) \pmod 3$. If $|D_3| > 3$, then the regular quadratic $\mathbb{F}_3$-space D_3 is universal, representing also elements $\frac{p}{3}$ with $p \not\equiv (2m+1) \pmod 3$. This is a contradiction. So $|D_3| = 1$ or $|D_3| = 3$. $\square$

Let Λ_+ be the even sublattice of $\Lambda = \langle X \rangle$. Then $\Lambda = \Lambda_+ \,\dot{\cup}\, \Lambda_-$ with $\Lambda_- = x + \Lambda_+$ for any $x \in X$. Since (x,y) is odd for all $x \in X$, the lattice

$$\Lambda_+ = \{ \sum_{x \in X} c_x x \mid c_x \in \mathbb{Z}, \sum_{x \in X} c_x \text{ even } \}$$

and $(\alpha, x) \in 2\mathbb{Z}$ for any $\alpha \in \Lambda_+$ and $x \in X$. Therefore $\Lambda_+ \subset 2\Lambda^*$ and the lattice $\Gamma := \frac{1}{\sqrt{2}} \Lambda_+$ is an integral lattice of dimension n.

The next lemma is an improvement of [4, Lemma 3.6].

Lemma 4.5 *Assume that $m(m+1)$ is not divisible by the square of an odd prime and that m is odd and $(m+1)$ is not a multiple of 8. Then for any $x \in X$*

$$\Gamma^*/\Gamma = \langle \frac{1}{\sqrt{2}} x + \Gamma \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. For odd primes p the Sylow p-subgroup of Γ^*/Γ is isomorphic to the one of Λ^*/Λ and hence $\{0\}$ for $p \geq 5$ and either $\{0\}$ or $\mathbb{Z}/3\mathbb{Z}$ for $p = 3$. Clearly $\alpha := \frac{1}{\sqrt{2}} x \in \Gamma^*$ has order 2 modulo Γ. Moreover

$$\Gamma^* = \sqrt{2}\Lambda_+^* = \langle \alpha, \sqrt{2}\Lambda^* \rangle$$

is an overlattice of $\sqrt{2}\Lambda^*$ of index 2. Now by Corollary 4.3 $(\beta,\beta) \in 2\mathbb{Z}_2$ for all elements $\beta \in \sqrt{2}\Lambda^*$ and since $x \in \Lambda$ we get $(\beta, \alpha) \in \mathbb{Z}$ for all $\beta \in \sqrt{2}\Lambda^*$. Since the Sylow 2-subgroup D_2 of Γ^*/Γ is a regular quadratic 2-group and $D_2 \cap \sqrt{2}\Lambda^*/\Gamma$ is in the radical of this group we obtain that $D_2 = \langle \alpha + \Gamma \rangle \cong \mathbb{Z}/2\mathbb{Z}$. To exclude the case that $D_3 = \mathbb{Z}/3\mathbb{Z}$ we use the fact that Γ is an even lattice and hence the Gauß sum

$$G(\Gamma) := \frac{1}{\sqrt{2 \cdot 3^t}} \sum_{d \in \Gamma^*/\Gamma} \exp(2\pi i q(d))$$

for the quadratic group $(\Gamma^*/\Gamma, q)$ with $q(z+\Gamma) := \frac{1}{2}(z,z) + \mathbb{Z}$ equals

$$G(\Gamma) = \exp(\frac{2\pi i}{8})^n = \exp(\frac{2\pi i}{8})^{-1}$$

by the Milgram-Braun formula. Clearly $G(\Gamma)$ is the product of the Gauß sums of its Sylow subgroups, $G(\Gamma) = G_2 G_3$ with

$$G_2 = \frac{1}{\sqrt{2}}(1 + \exp(2\pi i \frac{2m+1}{4})) = \frac{1-i}{\sqrt{2}} = \exp(\frac{2\pi i}{8})^{-1} = G(\Gamma)$$

since m is odd. This implies that $G_3 = 1$. Then [6, Corollary 5.8.3] shows that D_3 cannot be anisotropic, and hence by Lemma 4.4 $D_3 = \{0\}$. □

Theorem 4.6 *(see also [4, Theorem 3.10] for one case) Assume that $m(m+1)$ is not divisible by the square of an odd prime, m is even but not divisible by 8. Then $\Gamma^*/\Gamma \cong \mathbb{Z}/6\mathbb{Z}$ and $m \equiv -1 \pmod 3$.*

Proof. With the same proof as above we obtain $G(\Gamma) = \exp(\frac{2\pi i}{8})^{-1}$ and $G_2 = \exp(\frac{2\pi i}{8})$ and hence $G_3 = -i$. Then [6, Corollary 5.8.3] yields that $D_3 = \langle \beta + \Gamma \rangle$ with $3(\beta, \beta) \equiv 1 \pmod 3$. Let $\lambda := \sqrt{2}\beta \in \Lambda^*$. Then $(\lambda, \lambda) = \frac{p}{3}$ with $p \equiv 2 \pmod 3$. Then the integrality condition in Lemma 4.2 shows that

$$m(m+1)(2m-1) \in 9\mathbb{Z}_3$$

is a multiple of 9. This implies that $m \not\equiv 1 \pmod 3$ as it was already observed in [4] but also that $m \not\equiv 0 \pmod 3$. □

Corollary 4.7 $m \neq 3, 4, 6, 10, 12, 22, 28, 30, 34, 42, 46, \ldots$.

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